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Three charge supertubes in type-IIB plane wave backgrounds

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Abstract: We deform the supersymmetric black ring of five dimensional supergravity coupled to $N-1$ vector multiplets to obtain an asymptotically Gödel supersymmetric black ring. For the $U(1)^3$ model we lift this solution to obtain a three charge D1-D5-P supertube which asymptotes to a 1/2 supersymmetric plane wave of type-IIB supergravity. Further, we also show how one may deform the asymptotically flat three charge supertube of type-IIB, in the special case of vanishing KK dipole charge, to a three charge supertube which asymptotes to the maximally supersymmetric plane wave.

Keywords: Supergravity Models, Black Holes, Black Holes in String Theory, Penrose limit and pp-wave background.

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Contents

1. Introduction

Following the discovery of the vacuum rotating black ring[1] [1] a number of generalisations have been constructed. These could be organised broadly into two groups. The first encompasses asymptotically flat charged rotating rings, which include both the supersymmetric [3] and non-extremal black rings [4] of minimal supergravity as well as the dipole rings found in [5]. The second set consists of non-asymptotically flat solutions. It is relatively straightforward to use solution-generating techniques to construct black rings in fluxbranes [6 – 8]. More interesting has been the recent construction of supersymmetric black rings in Taub-NUT backgrounds [9 – 11]. Furthermore, Ortin has made use of the fact one can deform supersymmetric solutions of minimal supergravity in order to derive a black ring that asymptotes to the maximally supersymmetric Gödel spacetime [12]. In this note we show how one can trivially generalise this procedure to the more general case of minimal supergravity coupled to $N-1$ abelian vector multiplets. While these solutions are interesting in their own right, we will be interested in using them as a means to construct supertube configurations in ten and eleven dimensions.

One can lift solutions of the $\mathrm{U}(1)^3$ five dimensional supergravity to eleven dimensions. When one does this with the supersymmetric black ring solution, one obtains a black supertube. Via dimensional reduction and a series of certain T-dualities, they represent D1-D5-P supertubes in type-IIB supergravity [13 – 15]. On the other hand, performing this sequence of dualities on the Gödel solution, one obtains a supersymmetric plane wave [16]. Thus one would expect [12] that lifting Gödel black rings would lead to supertubes embedded in a plane wave. Indeed a three charge Gödel BMPV black hole was constructed in [17] using

[1] Although another rotating black ring has recently been found [2], due to its conical singularity it is strictly not a vacuum solution.

the reverse procedure. In that work, an asymptotically plane wave D1-D5-P configuration was constructed, which upon T-duality and dimensional reduction led to a three-charge Gödel BMPV black hole. In this note we construct three charge black supertubes that are asymptotically plane wave. We find that the near horizon geometry is unchanged by the deformation. These generalise the supertubes of [13]. We find the presence of the Kaluza-Klein monopole tube somewhat obstructs the inclusion of the plane wave. More precisely, if the Kaluza Klein dipole charge q^3 is non zero, our solutions necessarily contain closed timelike curves. Further, we can only construct a three charge supertube in the maximally supsersymmetric plane wave in the case where $q^3 = 0$.

This paper is organised as follows. First, we review the construction and supersymmetric solutions of the general five dimensional minimal supergravity coupled to $N-1$ abelian vector multiplets. We show how these can be deformed simply. Next, we uplift the solution and present the resulting type-IIB configuration. We show explicitly that far from the branes it asymptotes to a supersymmetric plane wave, supported by the appropriate Ramond-Ramond three form flux. In the following section we consider embedding the three charge supertubes in more general supersymmetric plane wave backgrounds. In particular we derive a solution with three charges and two dipole charges that asymptotes to the maximally supersymmetric plane wave. Finally, we conclude with a discussion and comment on extensions of the work.

2. Asymptotically Gödel supersymmetric black rings

To begin we will concern ourselves with $D = 5$ minimal supergravity coupled to $N-1$ abelian vector multiplets with scalars valued in a symmetric space. We follow the notation of [13]. The action of this theory is:

$$S = \frac{1}{16\pi}\int \left(R * 1 - G_{IJ} dX^I \wedge * dX^J - G_{IJ} F^I \wedge * F^J - \frac{1}{6} C_{IJK} F^I \wedge F^J \wedge A^K\right) \tag{2.1}$$

where $I, J, K = 1, \ldots, N$ and the scalars X^I are constrained by

$$\frac{1}{6} C_{IJK} X^I X^J X^K = 1 \tag{2.2}$$

where $C_{IJK} = C_{(IJK)}$ and the following condition is obeyed:

$$C_{IJK} C_{J'(LM} C_{PQ)K'} \delta^{JJ'} \delta^{KK'} = \frac{4}{3} \delta_{I(L} C_{MPQ)} \,. \tag{2.3}$$

The matrix G_{IJ} is defined as

$$G_{IJ} = \frac{9}{2} X_I X_J - \frac{1}{2} C_{IJK} X^K \tag{2.4}$$

where one lowers indices on the scalars as follows:

$$X_I = \frac{1}{6} C_{IJK} X^J X^K \,. \tag{2.5}$$

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The classification of supersymmetric solutions of this theory can be deduced from that of the gauged theory initiated in [18] and subsequently generalized in [19]. Such solutions admit a globally defined non-spacelike Killing vector field V. If there exists a neighbourhood where V is timelike one can choose coordinates (t, x^m) such that $V = \partial/\partial t$ and

$$ds^2 = -f^2(dt+\omega)^2 + f^{-1}h \tag{2.6}$$

where h is a Riemmannian metric on the base-space $\mathcal{B}$, f is a function and ω a 1-form both living on $\mathcal{B}$. Let $e^0 = f(dt+\omega)$ and we will choose an orientation of $\mathcal{B}$ such that $e^0 \wedge \mathrm{vol}(h)$ is positively oriented. We decompose $d\omega$ into self-dual and anti self-dual parts on the base as

$$f d\omega = G^+ + G^- \,. \tag{2.7}$$

Supersymmetry then implies that h is a hyper-Kähler metric on $\mathcal{B}$ and that

$$F^I = d(X^I e^0) + \Theta^I \tag{2.8}$$

where Θ^I are self-dual two forms on $\mathcal{B}$ such that

$$X_I \Theta^I = -\frac{2}{3} G^+ \,. \tag{2.9}$$

The Bianchi identities for F^I then give $d\Theta^I = 0$. The Maxwell equations imply

$$\nabla^2(f^{-1}X_I) = \frac{1}{6} C_{IJK} \Theta^J \cdot \Theta^K \,, \tag{2.10}$$

where $\alpha \cdot \beta \equiv \frac{1}{p!}\alpha^{m_1 m_2 \dots m_p}\beta_{m_1 m_2 \dots m_p}$ for p-forms α and β on $\mathcal{B}$.

Ortin has made the interesting observation that a solution of the minimal theory can be deformed by adding a piece to G^- while still leaving it a solution. This obviously generalises to the case considered here where we have a minimal theory coupled to $N-1$ vector multiplets. He showed that by a judicious choice of G^- one can make the supersymmetric ring asymptotically Gödel. We show here that the same construction works for the supersymmetric ring of the $\mathrm{U}(1)^N$ theory.

Firstly, we write down the $\mathrm{U}(1)^N$ supersymmetric black ring. The base $(\mathcal{B}, h)$ is $\mathbb{E}^4$. We write the metric in toroidal coordinates as

$$h = \sum_{i=1}^{4} dx_i^2 = \frac{R^2}{(x-y)^2}\left((y^2-1)d\psi^2 + \frac{dy^2}{y^2-1} + (1-x^2)d\phi^2 + \frac{dx^2}{1-x^2}\right). \tag{2.11}$$

The various quantities are given by

$$\Theta^I = -\frac{1}{2} q^I (dy \wedge d\psi + dx \wedge d\phi) \,, \tag{2.12}$$

$$f^{-3} = \frac{1}{6} C^{IJK} H_I H_J H_K \,, \tag{2.13}$$

$$\begin{aligned}\frac{1}{3} H_I \equiv f^{-1} X_I = \bar{X}_I + \frac{1}{6R^2}\left(Q_I - \frac{1}{2} C_{IJK} q^J q^K\right)(x-y) - \\ - \frac{1}{24R^2} C_{IJK} q^J q^K (x^2 - y^2) \,,\end{aligned} \tag{2.14}$$

$$
\begin{aligned}
\omega &= \omega_\phi d\phi + \omega_\psi d\psi\,, \\
\omega_\phi &= -\frac{1}{8R^2}(1-x^2)[q^I Q_I - \varrho(3+x+y)]\,, \\
\omega_\psi &= \frac{3}{2}(1+y)q^I \bar{X}_I - \frac{1}{8R^2}(y^2-1)[q^I Q_I - \varrho(3+x+y)]\,,
\end{aligned} \tag{2.15}
$$

where q^I, Q_I, $\bar{X}_I$ are constants, $C^{IJK} = C_{IJK}$ and $\varrho = \frac{1}{6}C_{IJK}q^I q^J q^K$. The constants $\bar{X}_I$ obey the same constraint as X_I do. The coordinate ranges are, as usual for black rings, $-1 \leq x \leq 1$ and $-\infty < y \leq -1$ and both angles have period 2π. There is an event horizon at $y = -\infty$. Using the method of Ortin, we deform the solution as follows. Simply replace ω by $\omega' = \omega + \omega_G$ where ω_G is given by

$$
\omega_G = \frac{\mu R^2}{(x-y)^2}[(1-x^2)d\phi - (y^2-1)d\psi]\,. \tag{2.16}
$$

It is easy to check that[2] $(d\omega_G)^+ = 0$ and thus G^+ for this deformed solution is the same as in the undeformed case. This is what implies it is still a solution. The remarkable fact is that this deformation leaves the horizon intact and thus the solution still represents a black ring. This is easy to see since as $y \to -\infty$ the extra terms arising in the metric from ω_G vanish. Also, as promised, the solution asymptotes to the maximally supersymmetric Gödel spacetime. To see this one needs to introduce the polar coordinates

$$
\rho \sin\theta = \frac{R\sqrt{y^2-1}}{x-y}\,, \qquad \rho\cos\theta = \frac{R\sqrt{1-x^2}}{x-y} \tag{2.17}
$$

where $0 \leq \theta \leq \pi/2$ and $0 \leq \rho < \infty$. Then using the fact that the undeformed solution is asymptotically flat it is easy to deduce that as $\rho \to \infty$

$$
ds^2 \sim -(dt + \omega_G)^2 + d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\psi^2 + \cos^2\theta d\phi^2)\,. \tag{2.18}
$$

In these coordinates $\omega_G = \mu\rho^2\sigma^3_R/2$ and $\sigma^3_R = d\phi' + \cos\theta' d\psi'$ is a right invariant form[3] on the S^3. This corresponds to the maximally supersymmetric Gödel solution. Note that this particular deformation does not introduce any Dirac-Misner strings ($\omega'_\phi(x = \pm 1) = \omega'_\psi(y = -1) = 0$) but closed timelike curves (CTC) will of course occur. We should note that the near-horizon limit of this deformed black ring is still locally $AdS_3 \times S^2$. This is easy to see using the technique of [13]. The extra terms present in the metric due to the deformation are

$$
-f^2\omega_G^2 - 2f^2(dt+\omega)\omega_G \tag{2.19}
$$

and as $y \to -\infty$, $f^2 \sim \text{const.} \times y^{-4}$, $\omega \sim \text{const.} \times y^3 d\psi$ and $\omega_G \sim -\mu R^2 d\psi$. Then letting[4] $\tilde{r} = -R^2/(\epsilon L y)$ and $\tilde{t} = t/\epsilon$ it is clear that as $\epsilon \to 0$ both of the above terms vanish. Note that the extra term present in each of the gauge fields also vanishes in this limit.

[2] The orientation is defined by $\epsilon_{y\psi x\phi} = +1$ as in [13] and corresponds to $\epsilon_{x_1x_2x_3x_4} = +1$.

[3] The Euler angles (θ', ϕ', ψ') are given by $\theta' = 2\theta$, $\psi' = \phi + \psi$ and $\phi' = \phi - \psi$.

[4] The radius of the S^1 of the ring at the horizon is denoted by L and is a function of the charges and R as given in [13].

The solution we have constructed can therefore be seen to describe asymptotically Gödel supersymmetric black rings coupled to $N-1$ vector multiplets. As in the undeformed case, they are $\frac{1}{2}$ BPS. We note in passing that in the $R=0$ limit one recovers the analogous supersymmetric Gödel BMPV black hole. This is made manifest by passing into the (ρ,θ) coordinate system.

3. Asymptotically plane wave D1-D5-P supertube

It is well known that $D = 11$ supergravity, reduced on a T^6 with coordinates z_a, $a = \{1,\ldots,6\}$, using the Ansatz

$$\begin{aligned} ds^2_{11} &= ds^2_5 + X^1(dz_1^2+dz_2^2) + X^2(dz_3^2+dz_4^2) + X^3(dz_5^2+dz_6^2) \\ C_3 &= A^1\wedge dz_1\wedge dz_2 + A^2\wedge dz_3\wedge dz_4 + A^3\wedge dz_5\wedge dz_6 \end{aligned} \tag{3.1}$$

yields the STU model. This has the action (2.1) with $N=3$, the only non vanishing component of C_{IJK} is $C_{123}=1$ and permutations, and the matrix $G_{IJ} = \frac{1}{2}(X^I)^{-2}\delta_{IJ}$. In this particular case, the solution ds^2_5 is given by (2.6) along with (2.11)–(2.15) and the replacement $\omega\to\omega'$, describes a $\frac{1}{2}$-BPS three-charge Gödel black ring. Note that the field strengths (2.8) are also deformed. Lifting these black rings to eleven dimensional supergravity then yields a straightforward deformation of the three-charge M-theory supertubes given in [13]. Explicitly, in terms of the H_I, we rewrite (2.6) as

$$ds^2_5 = -(H_1H_2H_3)^{-\frac{2}{3}}(dt+\omega')^2 + (H_1H_2H_3)^{\frac{1}{3}}\sum_{i=1}^{4}dx_i^2 \tag{3.2}$$

and the 1-form potentials are

$$A^I = H_I^{-1}(dt+\omega') - \frac{q^I}{2}((1+y)d\psi + (1+x)d\phi)\,. \tag{3.3}$$

Here, as in [13] we have set $\bar{X}_I = 1/3$. The four supercharges of the five dimensional solution are inherited to yield a $\frac{1}{8}$ BPS configuration. The undeformed system presented in [13] consists of three M2 branes carrying conserved charges proportional to the Q^I, and three M5 branes, each of which wrap the 'ring' ψ coordinate which is transverse to the membranes. As explained clearly in [13], these M5 branes do not carry conserved charges but instead possess 'dipole' charges parameterised by the q^I. It should be noted that the notion of mass for these objects is defined relative to an asymptotic minkowskian region transverse to the M2 branes. However, rather than being asymptotically flat, the solution presented here obviously asymptotes to the $\frac{5}{8}$ BPS supersymmetric Gödel universe that arises upon lifting (2.18).

To construct the type-IIB solution, Kaluza-Klein reduce on the compact direction z_6 and T-dualize along z_5, z_4, z_3. We could of course choose any of the z as the initial S^1. This choice corresponds to taking z_5 to point along the axis of the supertube. The resulting string frame metric is

$$ds^2_{\mathrm{IIB}} = -\frac{(dt+\omega')^2}{H_3\sqrt{H_1H_2}} + \frac{H_3}{\sqrt{H_1H_2}}(dz_5+A^3)^2 + \sqrt{H_1H_2}\sum_{i=1}^{4}dx_i^2 + \sqrt{\frac{H_2}{H_1}}\sum_{i=1}^{4}(dz_i)^2 \tag{3.4}$$

with dilaton

$$e^{2\Phi} = \frac{H_2}{H_1} \tag{3.5}$$

and three-form RR field strength

$$F_3 = (X^1)^{-2} \star_5 F^1 + F^2 \wedge (dz_5 + A^3) \,. \tag{3.6}$$

The solution above is similar to the D1-D5-P 'double-helix' supertube, except now it is not asymptotically flat. Given the fact that Gödel spacetimes are T-dual to plane waves, we expect something similar to manifest itself in this solution. To see this explicitly let us study the asymptotic form of this IIB solution. As $\rho \to \infty$

$$ds^2_{\text{IIB}} \sim dz_5^2 + 2dz_5(dt + \omega_G) + \sum_{i=1}^{4} dx_i^2 + \sum_{i=1}^{4} dz_i^2 \tag{3.7}$$

and if we let $Z = t + z_5$ we get

$$ds^2_{\text{IIB}} \sim -dt^2 + dZ^2 + 2(dZ - dt)\omega_G + \sum_{i=1}^{4} dx_i^2 + \sum_{i=1}^{4} dz_i^2 \,. \tag{3.8}$$

Note that this form of the metric is just as that found in [16] after T-dualising the Gödel IIA solution. Thus if one makes the coordinate transformation

$$u = t - Z \,, \qquad v = t + Z \tag{3.9}$$

$$\widetilde{\phi} = \phi - \mu u \,, \qquad \widetilde{\psi} = \psi + \mu u \,, \tag{3.10}$$

$$\widetilde{x}_1 + i\widetilde{x}_2 = r_1 e^{i\widetilde{\phi}} \,, \qquad \widetilde{x}_3 + i\widetilde{x}_4 = r_2 e^{i\widetilde{\psi}} \tag{3.11}$$

and noting $\rho^2 = r_1^2 + r_2^2$, the asymptotic form of the metric becomes

$$ds^2_{\text{IIB}} \sim -dudv - \mu^2 \left(\sum_{i=1}^{4} \widetilde{x}_i^2 \right) du^2 + \sum_{i=1}^{4} d\widetilde{x}_i^2 + \sum_{i=1}^{4} dz_i^2 \tag{3.12}$$

which is a 1/2 supersymmetric plane wave solution to type-IIB. The flux becomes

$$F_3 \sim -\frac{\mu}{2} du \wedge d(\rho^2 \widetilde{\sigma_R^3}) \,. \tag{3.13}$$

In fact it is the Penrose limit of $AdS_3 \times S^3 \times T^4$ supported by a three form RR-flux, which can be derived as the S-dual of $AdS_3 \times S^3 \times T^4$ supported by an NS-NS three form [24]. Hence the solution (3.4) seems to represent a D1-D5-P supertube in this plane wave background. One needs to check that this IIB solution has a regular horizon. Since we have noted that ds_5^2 is regular as $y \to -\infty$ we simply need to show that $dz_5 + A^3$ is also regular. One can do this in exactly the same manner as was done in [13] by performing the same shift in z_5 as they did. This is because the extra term we have in A^3 is $H_3^{-1}\omega_G$ which vanishes as $y \to -\infty$. In fact one can go further and show that the near-horizon limit of this IIB solution is unchanged by the deformation and thus is locally $AdS_3 \times S^3 \times T^4$. This can be deduced from the fact that ds_5^2 has the same near-horizon limit as the undeformed case together with the fact that the extra term in A^3 is $O(\bar{r}^2)$, where $\bar{r} = -R/y$. Thus the IIB solution we have constructed interpolates between $AdS_3 \times S^3 \times T^4$ and its Penrose limit.

A subtlety concerning the global structure of the spacetime derived here should be noted. Since the Kaluza-Klein direction z_5 is compact with period $2\pi R_z$ as in the undeformed solution, the lightcones coordinates $u = -z_5$ and $v = 2t + z_5$ inherit this periodicity. Indeed, enforcing the coordinate transformation (3.10) to be valid globally one deduces that $2\pi R_z \mu = 2\pi N$ for some integer N. Thus we see that the strength of the flux of the plane wave $\mu = N/R_z$ is quantized in units of inverse radius of the compact direction z_5. Asymptotically the deformed solution is actually a discrete quotient of a plane wave. Hence we will have CTC since $\partial/\partial u$ is timelike in this plane wave. It is not surprising that we have this situation, given that before the T-dualities we were dealing with an asymptotically Gödel spacetime which contains CTC at every point for sufficiently large radius. However in the IIB solution the CTC are milder in the sense that they are global, just like the ones encountered in AdS spacetimes. Moreover, these features are not specific to our solutions, but also occur for the D1-D5-P systems of [20, 17]. Curiously, one cannot pass to the covering space where u is a non-compact coordinate as z_5 has to be periodic when $q^3 > 0$ in order to avoid a Dirac-Misner string singularity at $x = 1$ [21]. Thus unlike the BMPV case we cannot remove these CTC, unless $q^3 = 0$ in which case the horizon is singular.

The microscopic derivation of the entropy of the asymptotically flat supertubes has been examined in [22], though no complete derivation has been given in terms of the D1-D5-P CFT. Moreover there has been success from the eleven dimensional standpoint by considering the CFT of the M5 brane intersection [23]. It is interesting to note that since the geometry of the horizon is unaffected by the deformation we introduced, the entropy of this configuration is the same as that of the asymptotically flat three charge supertubes found in [13]. Now, strings in the Penrose limit of $AdS_3 \times S^3 \times T^4$ can be easily quantized [24]. It turns out that the asymptotic density of states for this model is the same as in flat space (as $\alpha' \to 0$). This is nice as it suggests that given the microscopic derivation of the entropy of the three charge supertube in flat space one should find the same answer as for the three charge supertube in the plane wave solution constructed in this paper. This is consistent with the fact that the Bekenstein-Hawking entropy is unaffected by the plane wave as remarked above.

4. Supertubes in the maximally supersymmetric plane wave background

The solution we have constructed may be reduced along $S^1 \times T^4$ to yield the seed Gödel solution we started with. This of course requires the metric to be independent of the compactification coordinates. On the other hand, one may wish to consider supertube configurations embedded in more general plane wave backgrounds, such as the maximally supersymmetric plane wave background.

Consider the undeformed solution. The Einstein frame metric is given in terms of the string frame metric as $(g_E)_{\mu\nu} = e^{-\Phi/2}(g_S)_{\mu\nu}$. For the case at hand

$$ds_E^2 = H_1^{-1/4} H_2^{-3/4} \left[2(dt + \omega)(dz_5 + \Omega) + H_3 (dz_5 + \Omega)^2 \right] +$$

$$+H_1^{3/4}H_2^{1/4}\sum_{i=1}^{4}dx_i^2+\left(\frac{H_2}{H_1}\right)^{1/4}\sum_{i=1}^{4}dz_i^2\,, \tag{4.1}$$

$$\Omega\equiv-\frac{q^3}{2}((1+x)d\phi+(1+y)d\psi)\,. \tag{4.2}$$

Changing variables[5] to $2u=-z_5$ and $2v=2t+z_5$ transforms the terms in the square bracket to

$$-4dudv-2du(2\omega+\Omega)+(2dv+2\omega)\Omega+\Omega^2+(H_3-1)(2du-\Omega)^2\,. \tag{4.3}$$

This can be cast in a nicer form by introducing a new u coordinate by $du\to du+\Omega/2$ which leaves the full metric as

$$\begin{aligned} ds_E^2 &= H_1^{-1/4}H_2^{-3/4}\Big[-4dudv-2du(2\omega+\Omega)+4(H_3-1)du^2\Big]+\\ &\quad +H_1^{3/4}H_2^{1/4}\sum_{i=1}^{4}dx_i^2+\left(\frac{H_2}{H_1}\right)^{1/4}\sum_{i=1}^{4}dz_i^2\,. \end{aligned} \tag{4.4}$$

This is rather nice as we have cast the D1-D5-P supertube in exactly the same form as the D1-D5-P system which reduces to the BMPV black hole. This form of the metric allows one to add in extra pieces to the g_{uu} component very easily as was done for the BMPV system in [17]. This relies on the following observation:

$$ds^2 = e^{2A}\left(-4dudv+\mathcal{H}du^2+du\sum_{i=1}^{n}C_idx_i\right)+\sum_{i=1}^{n}e^{2B_i}dx_i^2 \tag{4.5}$$

$$\Rightarrow\quad R_{\mu\nu} = \bar{R}_{\mu\nu}-\frac{1}{2}\delta_\mu^u\delta_\nu^u e^{2A}\sum_{i=1}^{n}e^{-2B_i}[\partial_i^2\mathcal{H}+\partial_i\mathcal{H}\partial_iG_i+2(\partial_i^2A+\partial_iA\partial_iG_i)\mathcal{H}] \tag{4.6}$$

$$G_i = 2A-2B_i+\sum_{j=1}^{n}B_j \tag{4.7}$$

where A,B_i,C_i are all functions of the transverse coordinates x_i and $\bar{R}_{\mu\nu}$ is the Ricci tensor with $\mathcal{H}=0$.[6]

Armed with these results we now deform the D1-D5-P supertube as follows:

$$\begin{aligned} ds_E^2 &= H_1^{-1/4}H_2^{-3/4}\Big[-4dudv-2du(2\omega+\Omega)+(\mathcal{H}+4(H_3-1))du^2\Big]+\\ &\quad +H_1^{3/4}H_2^{1/4}\sum_{i=1}^{4}dx_i^2+\left(\frac{H_2}{H_1}\right)^{1/4}\sum_{i=1}^{4}dz_i^2\,. \end{aligned} \tag{4.8}$$

In view of the general result quoted above it is immediate that the the Ricci tensor of this deformed configuration is:

$$R_{\mu\nu}=\bar{R}_{\mu\nu}-\frac{1}{2H_1H_2}\delta_\mu^u\delta_\nu^u\left(\partial_{x_i}^2+H_1\partial_{z_i}^2-\frac{1}{4}\left(\partial_{x_i}^2\log H_1+3\partial_{x_i}^2\log H_2\right)\right)\mathcal{H}\,. \tag{4.9}$$

[5]Note that in this section our u,v coordinates are defined differently to the previous section, in order to match with [17].

[6]In [17] it was stated that the inclusion of angular momentum does not affect the result quoted in [20] which had $C_i=0$.

It should be noted that in this case the functions $G_i = 0$ and thus there are no terms with first derivatives of $\mathcal{H}$. We will choose to support this deformation by a five form flux as in [17]. Note that this is in contrast to the previous section where the deformation was supported by a three form flux. The five form flux must be self dual and $F_5 \wedge F_3 = 0$. It is natural to try and use the same expression as in [17]. Namely

$$F_5 = \mu du \wedge (dx_1 \wedge dx_2 \wedge dz_1 \wedge dz_2 + dx_3 \wedge dx_4 \wedge dz_3 \wedge dz_4)\,. \tag{4.10}$$

It is in fact straightforward to check that this five form is self dual also in this case. However in general $F_3 \wedge F_5 \neq 0$, see appendix. In the special case $q^3 = 0$ though $F_3 \wedge F_5 = 0$ and thus we may straightforwardly deform the solution in this situation. Note that the supertube is nakedly singular in this limit though. Nevertheless as noted in [13] the world volume theory of this configuration has a sensible interpretation in type-IIA in terms of D6 branes.

The analysis follows similarly to [17]. Explicitly, the type-IIB Einstein equations are

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu \Phi \partial_\nu \Phi + \frac{1}{96} F_{\mu\alpha\beta\gamma\delta} F_\nu{}^{\alpha\beta\gamma\delta} + \frac{e^\Phi}{4}\left(F_{\mu\alpha\beta} F_\nu{}^{\alpha\beta} - \frac{1}{12} g_{\mu\nu} F_3^2 \right). \tag{4.11}$$

Clearly only the uu component of the stress energy tensor is altered by the presence of $\mathcal{H}$. It is not difficult to check that these extra terms are

$$\Delta(F_{\mu\alpha\beta} F_\nu{}^{\alpha\beta}) = -2(\partial_i \log H_2)^2 H_1^{-\frac{1}{2}} H_2^{-\frac{3}{2}} \mathcal{H}\, \delta^u_\mu \delta^u_\nu \tag{4.12}$$

$$\Delta(g_{\mu\nu} F_3^2) = 6((\partial_i \log H_1)^2 - (\partial_i \log H_2)^2) H_1^{-\frac{1}{2}} H_2^{-\frac{3}{2}} \mathcal{H}\, \delta^u_\mu \delta^u_\nu \tag{4.13}$$

$$F_{\mu\alpha\beta\gamma\delta} F_\nu{}^{\alpha\beta\gamma\delta} = \frac{48\mu^2}{H_1 H_2} \delta^u_\mu \delta^u_\nu \tag{4.14}$$

where Δ represents the change in the quantity in the brackets due to the deformation $\mathcal{H}$. The equations of motion for the dilaton and F_3 are unchanged. Enforcing the uu component of the Einstein equations then leads to a simple linear equation for the deformation $\mathcal{H}$:

$$(\partial^2_{x_i} + H_1 \partial^2_{z_i})\mathcal{H} = -\mu^2\,. \tag{4.15}$$

Note that to do this one needs to use the fact that H_1, H_2 are harmonic when $q^3 = 0$. Thus the equation for the deformation is identical to that found in [17] for the BMPV system, except now H_1 is a harmonic function with delta function sources on a ring $\rho = R$ and $\theta = \pi/2$ as opposed to at the origin. This makes the PDE (4.15) more complicated and in fact if one writes it in (ρ, θ) or (x, y) coordinates it is not separable. Remarkably, if one uses yet a different coordinate system the equation can be made separable. The coordinates in question are:

$$r^2 = R^2 \frac{1-x}{x-y}\,, \qquad \cos^2\Theta = \frac{1+x}{x-y} \tag{4.16}$$

and were introduced in [13]. The flat space metric then looks like

$$\sum_{i=1}^4 dx_i^2 = \Sigma\left(\frac{dr^2}{r^2+R^2} + d\Theta^2 \right) + (r^2+R^2)\sin^2\Theta d\psi^2 + r^2 \cos^2\Theta d\phi^2 \tag{4.17}$$

where $\Sigma = r^2 + R^2 \cos^2\Theta$. It is immediately apparent that there are solutions of the form $\mathcal{H} = X(x_i) + Z(z_i)$ where $\partial^2_{z_i} Z = \alpha^2$ where α^2 is a separation constant. The resulting equation for X is then $\partial^2_{x_i} X + \alpha^2 H_1 = -\mu^2$, which upon multiplication by Σ is additively separable in the r, Θ coordinates system since $H_1 = 1 + Q_1/\Sigma$. This means that $X = F(r) + G(\Theta)$. The function $F(r)$ satisfies

$$\frac{1}{r}\frac{d}{dr}\left(r(r^2+R^2)\frac{dF}{dr}\right) + r^2(\alpha^2+\mu^2) = -\beta^2 \tag{4.18}$$

where β^2 is another separation constant. This may be be integrated to give

$$F(r) = -\frac{r^2}{8}(\alpha^2+\mu^2)+\frac{R^2}{8}(\alpha^2+\mu^2)\log(r^2+R^2)-\frac{\beta^2}{4}\log(r^2+R^2)+c_1\log\left(\frac{r}{\sqrt{r^2+R^2}}\right)+c_2 \tag{4.19}$$

where c_1, c_2 are integration constants. The equation for G may also be integrated, but it is convenient to change variables to $z = \sin^2\Theta$ first. In terms of z we have

$$4\frac{d}{dz}\left(z(1-z)\frac{dG}{dz}\right) + R^2(\alpha^2+\mu^2)(1-z) + Q_1\alpha^2 = \beta^2 \tag{4.20}$$

which leads to

$$G(z) = \frac{(Q_1\alpha^2-\beta^2)}{4}\log(1-z) + \frac{R^2(\alpha^2+\mu^2)}{8}(\log(1-z)-z) + c_3(\log z - \log(1-z)) + c_4\,. \tag{4.21}$$

We have generated quite a few constants upon integrating (4.15), however they may all be fixed as follows. The constants c_2, c_4 can be absorbed into shifts of v. Demanding regularity[7] at $\Theta = 0$ and $\pi/2$ forces $c_3 = 0$ and $\beta^2 = Q_1\alpha^2 + R^2(\alpha^2+\mu^2)/2$ respectively. Demanding regularity at $r = 0$ forces $c_1 = 0$ and finally requiring that $\mathcal{H} \sim \text{const.}(x_i x_i + z_i z_i)$ tells us that $\alpha^2 = -\mu^2/2$. Thus we arrive at

$$\mathcal{H} = -\frac{\mu^2}{16}(r^2 + R^2\sin^2\Theta + z_i z_i) + \frac{1}{8}Q_1\mu^2\log(r^2+R^2)\,. \tag{4.22}$$

This deformation gives a three charge, two dipole supertube which asymptotes to the maximally supersymmetric plane wave solution of type-IIB supergravity. The remarks concerning the global causal structure of these deformed supertube configurations given in the previous section also apply here, however in this case we do not get the quantization condition on μ. In particular, we can pass to the covering space where u is non-compact. One would expect in this case the solution to be devoid of CTC (given the constraint on the charges in [13]), though we have not performed a full analysis in the intermediate region (in between the near horizon and the asymptotic plane wave).

Finally we note in passing there is a class of solutions (with $\alpha = 0$) that are independent of the toroidal directions z_i; in this case, one could easily compactify on $S^1 \times T^4$ to derive nakedly singular asymptotically Gödel spacetimes in five dimensional minimal supegravity coupled to two vector multiplets.

[7]The horizon in these coordinates is located at $r = 0$ and $\Theta = \pi/2$, however as already noted it is not regular in the undeformed solution since $q^3 = 0$.

5. Concluding remarks

We have demonstrated how three charge supertubes can be embedded not only in asymptotically flat backgrounds, but also in the next simplest class of solutions, supersymmetric plane wave spacetimes. This was shown explicitly in two cases: firstly, for the Penrose limit of $AdS_3 \times S^3 \times T^4$, and secondly for the maximally supersymmetric BFHP plane wave. The former case was derived by first deforming the supersymmetric black rings in $D = 5$ minimal supergravity coupled to $N - 1$ vector multiplets. In the case $N = 3$, this solution was uplifted to eleven dimensions to describe three charge supertubes in the supersymmetric Gödel background. Upon Kaluza Klein reduction and T dualisation, a D1-D5-P supertube embedded in a $\frac{1}{2}$ BPS plane wave was constructed. Furthermore, the inclusion of the plane wave term does not affect the properties of the horizon. The removal of Dirac-Misner string singularities, however, leads to CTC, in contrast to the BMPV case.

In the second case, we use a more direct approach to derive a supertube plane wave configuration supported by a self dual five form flux. In order to satisfy the equations of motion, it seems one has to turn off the Kaluza-Klein dipole charge. By noting that the Ricci tensor has a simple decomposition under wave-like deformations of the metric we arrive at a simple linear PDE for the deformation. We show that this is additively separable, in suitable coordinates, and under this condition derive the general solution. A particular solution corresponds to supertubes in the maximally supersymmetric background.

There remain several open problems concerning asymptotically plane wave supertubes. Obviously, one might be interested in their world volume description. This could be particularly relevant for the solutions describing supertubes in the maximally supersymmetric background, as they seem to be nakedly singular from their supergravity description. Furthermore, one could try to generalise the solution presented here in the maximally supersymmetric plane wave background such that one has a non-zero Kaluza Klein dipole charge. Since these three charge, two dipole supertubes have non-extremal counterparts [4] with regular horizons, it might be useful to consider plane wave extensions of these thermally excited supertubes first.

Finally we should emphasise that it is most interesting that one can embed such supertube configurations in non-trivial backgrounds such as plane waves and Gödel spacetimes. These arose from lifting Gödel black rings. There remains, however, the interesting and apparently difficult problem of finding asymptotically AdS black rings.

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A. The three form F_3

The two form field strengths of the $\mathrm{U}(1)^N$ five dimensional supergravity are given by:

$$F^I = dA^I = d(X^I e^0) + \Theta^I = f^{-1} dH_I^{-1} \wedge e^0 + H_I^{-1} d\omega + \Theta^I \,. \tag{A.1}$$

It then follows that the Hodge dual is

$$*_5F^I = \frac{1}{3!}f^{-2}\partial_i H_I^{-1}\epsilon_{ilmn}dx^l \wedge dx^m \wedge dx^n + H_1^{-1}f^{-1}e^0 \wedge (G^+ - G^-) + e^0 \wedge \Theta^I \qquad \text{(A.2)}$$

where we have used the fact that $*_4 d\omega = f^{-1}(G^+ - G^-)$ and $*_4\Theta^I = \Theta^I$. Note that $\epsilon_{0x_1x_2x_3x_4} = 1$ has been used in these cartesian coordinates. The three form in IIB is given by

$$F_3 = (X^1)^{-2} *_5 F^1 + F^2 \wedge (dz_5 + A^3) = (X^1)^{-2} *_5 F^1 - 2F^2 \wedge du + F^2 \wedge (A^3 - \Omega)\,, \qquad \text{(A.3)}$$

where we have used the new u coordinate defined in section 4. Thus we will also need

$$F^2 \wedge (A^3 - \Omega) = f^{-1}H_2^{-1}H_3^{-1}d\omega \wedge e^0 + f^{-1}H_3^{-1}\Theta^2 \wedge e^0 \qquad \text{(A.4)}$$

$$= \frac{H_1^{1/3}}{H_2^{2/3}H_3^{2/3}}d\omega \wedge e^0 + \frac{H_1^{1/3}H_2^{1/3}}{H_3^{2/3}}\Theta^2 \wedge e^0 \qquad \text{(A.5)}$$

and

$$(X^1)^{-2} *_5 F^1 = \frac{1}{3!}f^{-2}(X^1)^{-2}\partial_i H_1^{-1}\epsilon_{lmni}dx^l \wedge dx^m \wedge dx^n + \\ + \frac{H_1^{1/3}}{H_2^{2/3}H_3^{2/3}}e^0 \wedge f^{-1}(G^+ - G^-) + (X^1)^{-2}e^0 \wedge \Theta^1\,. \qquad \text{(A.6)}$$

Upon adding A.5 to A.6 we see that the terms that look like $e^0 \wedge G^-$ cancel. Interestingly we find further cancellations. Using the identity $G^+ = -\frac{1}{2}fH_I\Theta^I$ leads to

$$\frac{H_1^{1/3}}{H_2^{2/3}H_3^{2/3}}2e^0 \wedge f^{-1}G^+ + (X^1)^{-2}e^0 \wedge \Theta^1 + f^{-1}H_3^{-1}e^0 \wedge \Theta^2 = -f^{-1}H_2^{-1}e^0 \wedge \Theta^3\,. \qquad \text{(A.7)}$$

Putting everything together the final explicit expression for F_3 is:

$$F_3 = -2F^2 \wedge du + \frac{1}{3!}f^{-2}(X^1)^{-2}\partial_i H_1^{-1}\epsilon_{ilmn}dx^l \wedge dx^m \wedge dx^n - f^{-1}H_2^{-2}e^0 \wedge \Theta^3\,. \qquad \text{(A.8)}$$

Thus as compared to the D1-D5-P BMPV system of [17] we have an extra term $-f^{-1}H_2^{-2} \times e^0 \wedge \Theta^3$. Note that this term is proportional to q^3, the KK dipole charge. Thus, shutting off the KK dipole charge of the supertube gets rid of this extra term leaving $F_3 \wedge F_5 = 0$ as required.

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